Infinite families of crossing-critical graphs
with prescribed average degree and crossing number

Drago Bokal[†]

Department of Mathematics
Institute of Mathematics, Physics, and Mechanics
Ljubljana, Slovenia
drago.bokal@imfm.uni-lj.si

LATEX-ed: September 14, 2009

Abstract

iráň constructed infinite families of k-crossing-critical graphs for every $k \geq 3$ and Kochol constructed such families of simple graphs for every $k \geq 2$. Richter and Thomassen argued that, for any given $k \geq 1$ and $r \geq 6$, there are only finitely many simple k-crossing-critical graphs with minimum degree r. Salazar observed that the same argument implies such a conclusion for simple k-crossing-critical graphs of prescribed average degree $r > 6$. He established existence of infinite families of simple k-crossing-critical graphs with any prescribed rational average degree $r \in [4, 6)$ for infinitely many k and asked about their existence for $r \in (3, 4)$. The question was partially settled by Pinontoan and Richter, who answered it positively for $r \in (3\frac{1}{2}, 4)$.

The present contribution uses two new constructions of crossing critical simple graphs along with the one developed by Pinontoan and Richter to unify these results and to answer Salazar's question by the following statement: for every rational number $r \in (3, 6)$ there exists an integer N_r, such that, for any $k > N_r$, there exists an infinite family of simple 3-connected crossing-critical graphs with average degree r and crossing number k. Moreover, a universal lower bound on k applies for rational numbers in any closed interval $I \subset (3, 6)$.

Keywords: crossing number, critical graph, crossing-critical graph, average degree, graph.

1 Introduction

Let $\mathrm{cr}(G)$ denote the crossing number of a graph G. A graph G is *k-crossing-critical*, if $\mathrm{cr}(G) \geq k$ and $\mathrm{cr}(G - e) < k$ for any edge $e \in E(G)$. Note that unless stated otherwise, all graphs in this paper are without vertices of degree two, as such vertices are trivial with respect to crossing number. The graphs may contain multiple edges, but do not contain loops. Besides that, the standard terminology from [7] is used.

Crossing-critical graphs give insight into structural properties of the crossing number invariant and have thus generated considerable interest. iráň introduced *crossing-critical edges* and proved that any such edge e of a graph G with $\mathrm{cr}(G - e) \leq 1$ belongs to a Kuratowsky subdivision in G [24]. Moreover, such a claim does not hold for edges with $\mathrm{cr}(G - e) \geq 5$. In [25], iráň constructed the first infinite family of 3-connected k-crossing-critical graphs for arbitrary given $k \geq 3$. Kochol constructed the first infinite family of simple 3-connected k-crossing-critical graphs ($k \geq 2$) in [15]. Richter and Thomassen proved that $\mathrm{cr}(G) \leq \frac{5}{2}k + 16$ for a k-crossing-critical graph G in [19]. They used this result to prove that there are only finitely many simple k-crossing-critical graphs with minimum degree r for any integers $k \geq 1$ and $r \geq 6$ and constructed an infinite family of simple 4-regular 4-connected 3-crossing-critical graphs and posed a question about existence of simple 5-regular k-crossing-critical graphs. Salazar observed that their argument implies finiteness of the number of the number of simple k-crossing critical graphs of average degree r for any rational $r > 6$ and integer $k > 0$ [21]. Since the finiteness of the set of simple 3-regular k-crossing-critical graphs can be established using Robertson-Seymour graph minor theory, it follows that the only average degrees for which an infinite family of simple k-crossing-critical graphs could exist are $r \in (3, 6]$. Salazar constructed an infinite family of simple k-crossing-critical graphs with average degree r for any $r \in [4, 6)$ and posed the following question:

Question 1 ([21]) *Let r be a rational number in $(3, 4)$. Does there exist an integer k and an infinite family of (simple) graphs, each of which has average degree r and is k-crossing-critical?*

Question 1 was partially answered by Pinontoan and Richter [18]. They proposed constructing crossing-critical graphs from smaller pieces or tiles, and applied this idea to design infinite families of simple k-crossing-critical graphs for any prescribed average degree $r \in (3\frac{1}{2}, 4)$.

Besides the study of degrees in crossing-critical graphs, there are also some structural results. Salazar improved the factor $\frac{5}{2}$ in the bound of Richter and Thomassen to 2 for large k-crossing-critical graphs [22] and for graphs of minimum degree four [23]. Hliněný proved that there is a function f such that no k-crossing-critical graph contains a subdivision of a binary tree of height $f(k)$, which implies that the path-width of such a graph is at most $2^{f(k)+1} - 2$. In particular, $k - 1 \leq f(k) \leq 6(72 \log_2 k + 248)k^3$ [10, 11]. Existence of a bound on the path-width of k-crossing-critical graphs was first conjectured by Geelen, Richter, Salazar, and Thomas in [9], where they established a result implying a bound on the tree-width of k-crossing-critical graphs. Hliněný defined crossed k-fences, which are k-crossing-critical graphs, in [10]. Crossed k-fences from some particular family contain subdivisions of binary trees of height $k - 1$ and thus have path-width at least $2^k - 2$.

Focus of the research on crossing-critical graphs was on 3-(edge)-connected crossing-critical graphs. This condition eliminates vertices of degree two, which are trivial with respect to the crossing number. But the condition is much stronger and its application has been justified only recently by a structural result of Leaños and Salazar in [16], stating that, for a connected crossing-critical graph G with minimum degree at least three, there exists a collection $G_1, \ldots, G_m$ of 3-edge-connected crossing-critical graphs, each of which is contained as a subdivision in G, and such that $\mathrm{cr}(G) = \sum_{i=1}^{m} \mathrm{cr}(G_i)$.

Two new constructions of crossing-critical graphs are developed in this contribution. In combination with the one of Pinontoan and Richter [18], they are applied to answer the

question of Salazar by a result resembling those of iráň and Kochol: we show that there exist infinite families of simple k-crossing-critical graphs with any prescribed average degree $r \in (3,6)$, for any k greater than some lower bound N_r. This leaves average degree $r = 6$ as the only open case. Several steps are required for our proof. In Section 2, the theory of tiles of Pinontoan and Richter is extended to yield effective lower bounds on the number of tiles needed to imply the lower bounds on crossing number. Section 3 contains the first new construction of k-crossing critical graphs, which yields infinite families of such graphs with average degree arbitrarily close to three. The second new construction relies on a sufficient condition that the zip product, studied in [1, 2], preserves criticality of the graphs involved. This is established in Section 4. The main result is proved in Section 5 by combining the results of the previous sections. Some further aspects of applying zip product in construction of crossing-number critical graphs are discussed as the conclusion in Section 6.

2   Tiles

In this section, we present a variant of the theory of tiles developed by Pinontoan and Richter [18]. In particular, we consider general sequences of not necessarily equal tiles, avoid the condition that the tiles be connected, and allow forming double edges when joining tiles. Such generalizations do not hinder the arguments of [18] and are useful in further investigations of tiled graphs. We establish an effective bound on the number of tiles needed to imply lower bounds on crossing numbers. Finally, we combine these improvements into a general construction of crossing-critical graphs.

Let G be a graph and $\lambda = (\lambda_0, \ldots, \lambda_l)$, $\rho = (\rho_0, \ldots, \rho_r)$ two sequences of distinct vertices, such that no vertex of G appears in both. The triple $T = (G, \lambda, \rho)$ is called a *tile*. To simplify the notation, we may sometimes use T in place of its graph G and we may consider sequences λ and ρ as sets of vertices. For $u, v \in \lambda$ or $u, v \in \rho$, we use $u \leq v$ or $u \geq v$ whenever u precedes or succeeds v in the respective sequence.

A drawing of G in the unit square $[0,1] \times [0,1]$ that meets the boundary of the square precisely in the vertices of the *left wall* λ, all drawn in $\{0\} \times [0,1]$, and the *right wall* ρ, all drawn in $\{1\} \times [0,1]$, is a *tile drawing* of T if the sequence of decreasing y-coordinates of the vertices of each λ and ρ respects the corresponding sequence λ or ρ. The *tile crossing number* $\mathrm{tcr}(T)$ of a tile T is the minimum number of crossings over all tile drawings of T.

Let $T = (G, \lambda, \rho)$ and $T' = (G', \lambda', \rho')$ be two tiles. We say that T is *compatible* with T' if $|\rho| = |\lambda'|$. A tile T is *cyclically-compatible* if it is compatible with itself. A sequence of tiles $\mathcal{T} = (T_0, \ldots, T_m)$ is *compatible* if T_i is compatible with T_{i+1} for $i = 0, \ldots, m-1$. It is *cyclically-compatible* if it is compatible and T_m is compatible with T_0. All sequences of tiles are assumed to be compatible.

The *join* of two compatible tiles T and T' is defined as $T \otimes T' = (G \otimes G', \lambda, \rho')$, where $G \otimes G'$ is the graph obtained from the disjoint union of G and G' by identifying ρ_i with λ'_i for $i = 0, \ldots, |\rho| - 1$. This operation is associative, thus we can define the *join* of a compatible sequence of tiles $\mathcal{T} = (T_0, \ldots, T_m)$ to be the tile $\otimes \mathcal{T} = T_0 \otimes T_1 \otimes \ldots \otimes T_m$. Note that we may produce multiple edges or vertices of degree two when joining tiles. We keep the double edges, but remove the vertices of degree two by contracting one of the incident edges.

For a cyclically-compatible tile $T = (G, \lambda, \rho)$, we define its *cyclization* $\circ T$ as the graph, obtained from G by identifying λ_i with ρ_i for $i = 0, \ldots, |\rho| - 1$. Similarly, we define the cyclization of a cyclically-compatible sequence of tiles as $\circ \mathcal{T} = \circ(\otimes \mathcal{T})$.

Lemma 2 ([18]) *Let T be a cyclically-compatible tile. Then, $\mathrm{cr}(\circ T) \leq \mathrm{tcr}(T)$. Let $\mathcal{T} = (T_0, \ldots, T_m)$ be a compatible sequence of tiles. Then, $\mathrm{tcr}(\otimes \mathcal{T}) \leq \sum_{i=0}^{m} \mathrm{tcr}(T_i)$.*

For a sequence ω, let $\bar{\omega}$ denote the reversed sequence. For a tile $T = (G, \lambda, \rho)$, let its *right-inverted* tile $T^{\updownarrow}$ be the tile $(G, \lambda, \bar{\rho})$, its *left-inverted* tile $^{\updownarrow}T$ be the tile $(G, \bar{\lambda}, \rho)$, and its *inverted* tile be the tile $^{\updownarrow}T^{\updownarrow} = (G, \bar{\lambda}, \bar{\rho})$. The *reversed* tile of T is the tile $T^{\leftrightarrow} = (G, \rho, \lambda)$.

Let $\mathcal{T} = (T_0, \ldots, T_m)$ be a sequence of tiles. A *reversed* sequence of $\mathcal{T}$ is the sequence $\mathcal{T}^{\leftrightarrow} = (T_m^{\leftrightarrow}, \ldots, T_0^{\leftrightarrow})$. A *twist* of $\mathcal{T}$ is the sequence $\mathcal{T}^{\updownarrow} = (T_0, \ldots, T_{m-1}, T_m^{\updownarrow})$. Let $i \in \{0, \ldots, m\}$ be arbitrary. Then, an *i-flip* of $\mathcal{T}$ is the sequence $\mathcal{T}^i = (T_0, \ldots, T_{i-1}, T_i^{\updownarrow}, {}^{\updownarrow}T_{i+1}, T_{i+2}, \ldots, T_m)$, an *$i$-cut* of $\mathcal{T}$ is the sequence $\mathcal{T}/i = (T_{i+1}, \ldots, T_m, T_0, \ldots, T_{i-1})$, and an *$i$-shift* of $\mathcal{T}$ is the sequence $\mathcal{T}_i = (T_i, \ldots, T_m, T_0, \ldots, T_{i+1})$. For the last two operations, cyclic compatibility of $\mathcal{T}$ is required.

Two sequences of tiles $\mathcal{T}$ and $\mathcal{T}'$ of the same length m are *equivalent* if one can be obtained from the other by a sequence of shifts, flips, and reversals. It is easy to see that the graphs $\circ \mathcal{T}$ and $\circ \mathcal{T}'$ are equal for equivalent cyclically-compatible sequences $\mathcal{T}$ and $\mathcal{T}'$ and thus have the same crossing number.

We say that a tile $T = (G, \lambda, \rho)$ is *planar* if $\mathrm{tcr}(T) = 0$ holds. It is *connected* if G is connected. It is *perfect* if:

(p.i) $|\lambda| = |\rho|$,

(p.ii) both graphs $G - \lambda$ and $G - \rho$ are connected,

(p.iii) for every $v \in \lambda$ or $v \in \rho$ there is a path from v to a vertex in ρ (λ) in G internally disjoint from λ (ρ), and

(p.iv) for every $0 \leq i < j \leq |\lambda|$ there is a pair of disjoint paths P_{ij} and P_{ji} in G, such that P_{ij} joins λ_i with ρ_i and P_{ji} joins λ_j with ρ_j.

Note that perfect tiles are connected.

Lemma 3 ([18]) *For a cyclically-compatible perfect planar tile T and a compatible sequence $\mathcal{T} = (T_0, \ldots, T_m, T)$, there exists $n \in \mathbb{N}$, such that, for every $k \geq n$, $\mathrm{tcr}((\otimes \mathcal{T}) \otimes (T^k)) = \mathrm{tcr}((\otimes \mathcal{T}) \otimes (T^n))$.*

Let $T = (G, \lambda, \rho)$ be a tile and H a graph that contains G as a subgraph. The *complement* of the tile T in H is the tile $H - T = (H[(V(H) \setminus V(G)) \cup \lambda \cup \rho] - E(G), \rho, \lambda)$. We can consider it as the edge complement of the subgraph G of H from which we remove all the vertices of T not in its walls. Whenever $\circ(T \otimes (H - T)) = H$, i.e. if the vertices of $\lambda \cup \rho$ separate G from $H - G$, we say that T is a *tile in H*. Using this concept, the following lemma shows the essence of perfect tiles.

4

Lemma 4 *Let $T = (G, \lambda, \rho)$ be a perfect planar tile in a graph H, such that there exist two disjoint connected subgraphs G_λ and G_ρ of H contained in the same component of $H - T$ and with $G \cap G_\lambda = (\lambda, \emptyset)$, $G \cap G_\rho = (\rho, \emptyset)$. If $E(G)$ and either $E(G_\lambda)$ or $E(G_\rho)$ are not crossed in some drawing D of H, then the D-induced drawings of T and its complement $H - T$ are homeomorphic to tile drawings.*

Proof. There is only one component of $H - T$ containing the vertices of $\lambda \cup \rho$, and as the edges of other components do not cross G nor influence its induced drawing, we may assume that $H - T$ *is* that component and, in particular, it is connected.

Denote by D_T the D-induced drawing of T, by T^- the tile $H - T$, and by D^- the D-induced drawing of T^-. As the edges of T are not crossed in D and T^- is connected, there is a face F of D_T containing D^-. The boundary of F contains all vertices of $T \cap T^- = \lambda \cup \rho$. Let W be the facial walk of F. No vertex of $\lambda \cup \rho$ appears twice in W: such a vertex would be a cutvertex in the planar graph G. Then either $G - \lambda$ or $G - \rho$ would not be connected, violating (p.ii), or some vertex in $\lambda \cup \rho$ would have no path to the opposite wall, as required by (p.iii).

Let W' be the induced sequence of vertices of $\lambda \cup \rho$ in W. As the edges of G_λ or G_ρ are not crossed in D and T, G_λ, and G_ρ are connected, the vertices of λ do not interlace with the vertices of ρ in W'. The ordering of λ in W' is the inverse ordering of ρ in W', since the disjoint paths from (p.iv) do not cross in D_T. The planarity and the connectedness of T imply that whenever $i < j < l$ or $i > j > l$, there is a path Q from P_{jl} to λ_i disjoint from P_{lj}. Q does not cross P_{lj} in D_T, thus $W' = \lambda\bar{\rho}$ or $W' = \rho\bar{\lambda}$. The claim follows. $\qquad\square$

The above arguments were in [18] combined with Lemma 3 to demonstrate the following:

Theorem 5 ([18]) *Let T be a perfect planar tile and let $\bar{T}_k = T^k \otimes T^\updownarrow \otimes T^k$ for $k \geq 1$. Then there exist integers n, N, such that $\mathrm{cr}(\circ(\bar{T}_k)) = \mathrm{tcr}(\bar{T}_n)$ for every $k \geq N$.*

We establish effective values of n and N from the above theorem:

Theorem 6 *Let $\mathcal{T} = (T_0, \ldots, T_l, \ldots, T_m)$ be a cyclically-compatible sequence of tiles. Assume that, for some integer $k \geq 0$, the following hold: $m \geq 4k - 2$, $\mathrm{tcr}(\otimes \mathcal{T}/i) \geq k$, and the tile T_i is a perfect planar tile, both for every $i = 0, \ldots, m$, $i \neq l$. Then, $\mathrm{cr}(\circ \mathcal{T}) \geq k$.*

Proof. We may assume $k \geq 1$. Let $G = \circ \mathcal{T}$ and let D be an optimal drawing of G. Assume that D has less than k crossings. Then there are at most $2k - 1$ tiles in the set $\mathcal{S} = \{T_i \mid i = l \text{ or } E(T_i) \text{ crossed in } D\}$. The circular sequence $\mathcal{T}$ is by the tiles of $\mathcal{S}$ fragmented into at most $2k - 1$ segments. By the pigeon-hole principle, the set $\mathcal{T} \setminus \mathcal{S}$, which consists of at least $2k$ tiles, contains two consecutive tiles $T_i T_{i+1}$. Assume for simplicity that $i = 1$, then either T_0 or T_3 is distinct from T_l. Lemma 4 with (G, T_1, T_0, T_2) or (G, T_2, T_1, T_3) in place of $(H, T, G_\lambda, G_\rho)$ establishes that the induced drawing D^- of $G - T_j$ is a tile drawing for some $j \in \{1, 2\}$. Since D^- contains all the crossings of D, this contradicts $\mathrm{tcr}(\otimes(\mathcal{T}/j)) \geq k$, and the claim follows. $\qquad\square$

Corollary 7 *Let $\mathcal{T} = (T_0, \ldots, T_l, \ldots, T_m)$ be a cyclically-compatible sequence of tiles and $k = \min_{i \neq l} \mathrm{tcr}(\otimes \mathcal{T}/i)$. If $m \geq 4k - 2$ and the tile T_i is a perfect planar tile for every $i = 0, \ldots, m$, $i \neq l$, then $\mathrm{cr}(\circ \mathcal{T}) = k$.*

5

Proof. By Lemma 2 and the planarity of tiles, $\text{cr}(\circ\mathcal{T}) \leq \text{tcr}((\otimes\mathcal{T}/i) \otimes T_i) \leq \text{tcr}(\otimes\mathcal{T}/i)$ for any $i \neq l$, thus $\text{cr}(\circ\mathcal{T}) \leq k$. Theorem 6 establishes k as a lower bound and the claim follows. $\square$

A tile T is *k-degenerate* if it is perfect, planar, and $\text{tcr}(T^{\updownarrow} - e) < k$ for any edge $e \in E(T)$. A sequence of tiles $\mathcal{T} = (T_0, \ldots, T_m)$ is *k-critical* if the tile T_i is k-degenerate for every $i = 0, \ldots, m$ and $\min_{i \neq m} \text{tcr}(\otimes(\mathcal{T}^{\updownarrow}/i)) \geq k$. Note that $\text{tcr}(T^{\updownarrow}) \geq k$ for every tile T in a k-critical sequence.

Corollary 8 *Let $\mathcal{T} = (T_0, \ldots, T_m)$ be a k-critical sequence of tiles. Then, $T = \otimes\mathcal{T}$ is a k-degenerate tile. If $m \geq 4k - 2$ and $\mathcal{T}$ is cyclically-compatible, then $\circ(T^{\updownarrow})$ is a k-crossing-critical graph.*

Proof. Lemma 2 implies that T is a planar tile. By induction it is easy to show that T is a perfect tile. Let e be an edge of T and let i be such that $e \in T_i$. The sequence $\mathcal{T}' = (T_0, \ldots, T_{i-1}, T_i^{\updownarrow}, {}^{\updownarrow}T_{i+1}{}^{\updownarrow}, \ldots, {}^{\updownarrow}T_m^{\updownarrow})$ is equivalent to $\mathcal{T}^{\updownarrow}$. Lemma 2 establishes $\text{tcr}(T^{\updownarrow} - e) = \text{tcr}((\otimes\mathcal{T}') - e) \leq \text{tcr}(T_i^{\updownarrow} - e) < k$, thus T is a k-degenerate tile.

Let $\mathcal{T}$ be cyclically-compatible. Then $\text{cr}((\circ T^{\updownarrow}) - e) < k$ for any edge $e \in E(T)$. Theorem 6 implies $\text{cr}(\circ(T^{\updownarrow})) \geq k$ for $m \geq 4k - 2$. Thus, $\circ(T^{\updownarrow})$ is a k-crossing-critical graph. $\square$

The above results provide sufficient conditions for the crossing numbers of certain graphs to be estimated in terms of the tile crossing numbers of their subgraphs. In what follows, we develop some techniques to estimate the tile crossing number.

A general tool we employ for this purpose is the concept of a *gadget*. We do not define it formally; a gadget can be any structure inside a tile $T = (G, \lambda, \rho)$, which guarantees a certain number of crossings in every tile drawing of T. Pinontoan and Richter used twisted pairs as gadgets [18], and we present staircase strips. Some other possible gadgets are cloned vertices, which were already used by Kochol [15], wheel gadgets, and others, which were studied in [3].

In general, there can be many gadgets inside a single tile. Whenever they are edge disjoint, the crossings they force in tile drawings are distinct. The following weakening of disjointness enables us to prove stronger results. For clarity, we first state the condition in its set-theoretic form.

Let A_1, B_1, A_2, B_2 be four sets. The unordered pairs $\{A_1, B_1\}$ and $\{A_2, B_2\}$ are *coherent* if one of the sets X_i, $X \in \{A, B\}$, $i \in \{1, 2\}$, is disjoint from $A_{3-i} \cup B_{3-i}$.

Lemma 9 *Let $\{A, B\}$ and $\{A', B'\}$ be two pairs of sets. If they are coherent and*

$$a \in A, b \in B, a' \in A' \text{ and } b' \in B', \tag{2.1}$$

then the unordered pairs $\{a, b\}$ and $\{a', b'\}$ are distinct. Conversely, if (2.1) implies distinctness of $\{a, b\}$, $\{a', b'\}$ for every quadruple a, b, a', b', then the pairs $\{A, B\}$, $\{A', B'\}$ are coherent.

Proof. Suppose the pairs are not distinct, then either $a = a'$ and $b = b'$, or $a = b'$ and $b = a'$. In both cases, every set has a member in the union of the other pair, and the pairs are not coherent.

For the converse, suppose the pairs would not be coherent. Then every set would contain an element in the union of the opposite pair. Let $x \in A \cap A'$, assuming the intersection is not empty. If there is an element $y \in B' \cap B$, then the quadruple $a = x$, $b = y$, $a' = x$, $b' = y$ satisfies (2.1) but does not form two distinct pairs. If $B \cap B'$ is empty, then there must be $a' \in B \cap A'$ and $b' \in B' \cap A$. The quadruple $a = a'$, $b = b'$, a', b' satisfies (2.1). Assuming $x \in A \cap B'$, a similar analysis applies and the claim follows. $\square$

Lemma 9 has an immediate application to crossings: whenever the pairs of edges $\{e_x, f_x\}$ and $\{e_y, f_y\}$ are distinct for two crossings x and y, the crossings x and y are distinct. Distinctness of crossings induced by two coherent pairs of sets of edges in a graph follows.

The notion of coherence can be generalized. Let $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_n\}$ be two families of sets. They are *coherent* if the two pairs $\{A_i, A_j\}$ and $\{B_k, B_l\}$ are coherent for every $0 \leq i < j \leq m$, $0 \leq k < l \leq n$.

A path P in G is a *traversing path* in a tile $T = (G, \lambda, \rho)$ if there exist indices $i(P) \in \{0, \ldots, |\lambda| - 1\}$ and $j(P) \in \{0, \ldots, |\rho| - 1\}$ such that P is a path from $\lambda(P) = \lambda_{i(P)}$ to $\rho(P) = \rho_{j(P)}$ and $\lambda(P)$, $\rho(P)$ are the only wall vertices that lie on P. An (unordered) pair of disjoint traversing paths $\{P, Q\}$ is *aligned* if $i(P) < i(Q)$ is equivalent to $j(P) < j(Q)$, and *twisted* otherwise. Disjointness of the traversing paths in a twisted pair $\{P, Q\}$ implies that some edge of P must cross some edge of Q in any tile drawing of T. Two pairs $\{P, Q\}$ and $\{P', Q'\}$ of traversing paths in T are *coherent* if $\{E(P), E(Q)\}$ and $\{E(P'), E(Q')\}$ are coherent. A family of pairwise coherent twisted (respectively, aligned) pairs of traversing paths in a tile T is called a *twisted (aligned)* family in T.

Lemma 10 ([18]) *Let $\mathcal{F}$ be a twisted family in a tile T. Then, $\mathrm{tcr}(T) \geq |\mathcal{F}|$.*

Let a tile T be compatible with T' and let $\{P, Q\}$ be a twisted pair of traversing paths of T. An aligned pair $\{P', Q'\}$ of traversing paths in T' *extends* $\{P, Q\}$ to the right if $j(P) = i(P')$, $j(Q) = i(Q')$. Then $\{PP', QQ'\}$ is a twisted pair in $T \otimes T'$. For a twisted family $\mathcal{F}$ in T, a *right-extending family* is an aligned family $\mathcal{F}'$ in T', for which there exists a bijection $e : \mathcal{F} \to \mathcal{F}'$, such that the pair $e(\{P, Q\}) \in \mathcal{F}'$ extends the pair $\{P, Q\}$ on the right. In this case, the family $\mathcal{F} \otimes_e \mathcal{F}' = \{\{PP', QQ'\} \mid \{P', Q'\} = e(\{P, Q\})\}$ is a twisted family in $T \otimes T'$. Extending to the left is defined similarly. Let $\mathcal{T} = (T_0, \ldots, T_l, \ldots, T_m)$ be a compatible sequence of tiles and $\mathcal{F}_l$ a twisted family in T_l. If, for $i = l + 1, \ldots, m$ (respectively, $i = l - 1, \ldots, 0$), there exist aligned right- (left-) extending families $\mathcal{F}_i$ of $\mathcal{F}_l \otimes \ldots \otimes \mathcal{F}_{i-1}$ ($\mathcal{F}_{i+1} \otimes \ldots \otimes \mathcal{F}_{l-1}$), then $\mathcal{F}_l$ *propagates* to the right (left) in $\mathcal{T}$. $\mathcal{F}_l$ *propagates* in cyclically-compatible $\mathcal{T}$ if it propagates both to the left and to the right in every cut $\mathcal{T}/i$, $i = 0, \ldots, m$, $i \neq l$.

A twisted family $\mathcal{F}$ in a tile T *saturates* T if $\mathrm{tcr}(T) = |\mathcal{F}|$, i.e. there exists a tile drawing of T with $|\mathcal{F}|$ crossings. Clearly, all these crossings must be on the edges of pairs of paths in $\mathcal{F}$.

Corollary 11 *Let $\mathcal{T} = (T_0, \ldots, T_l, \ldots, T_m)$ be a cyclically-compatible sequence of tiles and $\mathcal{F}$ a twisted family in T_l that propagates in $\mathcal{T}$. If $m \geq 4|\mathcal{F}| - 2$ and the tile T_i is a perfect planar tile for every $i = 0, \ldots, m$, $i \neq l$, then $\mathrm{cr}(\circ\mathcal{T}) \geq |\mathcal{F}|$. If $\mathcal{F}$ saturates T_l, then the equality holds.*
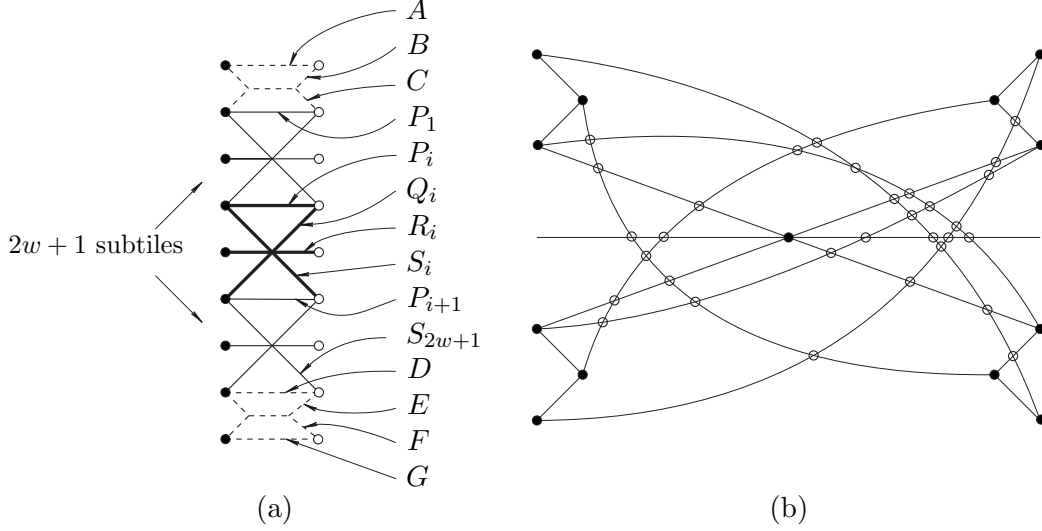
7

Figure 1: (a) The tile H_w, $w = 1$. (b) An optimal tile drawing of H_0.

Proof. As $\mathcal{F}$ propagates in $\mathcal{T}$, Lemma 10 implies $\min_{i \neq l} \mathrm{tcr}(\otimes(\mathcal{T}/i)) \geq |\mathcal{F}|$. Theorem 6 establishes the claim. $\square$

Let H_w be a tile, which is for $w = 1$ presented in Figure 1 (a). It is constructed by joining two subtiles, denoted by dashed edges, with a sequence of $2w + 1$ subtiles, of which one is drawn with thick edges. The left (right) wall vertices of H_w are colored black (white). H_w is a perfect planar tile. Let $\mathcal{H}(w, s) = (H_w, \ldots, H_w)$ be a sequence of tiles of length s and let $H(w, s) = \circ(\mathcal{H}(w, s)^{\updownarrow})$ be the cyclization of its twist.

Proposition 12 *The graph $H(w, s)$ is a crossing-critical graph with crossing number $k = 32w^2 + 56w + 31$ whenever $s \geq 4k - 1$.*

Proof. Using the traversing paths $A, \ldots, G$ of H_w depicted in Figure 1, we construct an aligned family $\mathcal{H}_w$ of size k, cf. [3, 4]. The corresponding family $\mathcal{H}'_w$ in $H_w^{\updownarrow}$ is twisted and propagates in $\mathcal{H}(w, s)^{\updownarrow}$. Figure 1 (b) presents an optimal tile drawing of H_0, its generalization to $w > 0$ demonstrates that $\mathcal{H}'_w$ saturates $H_w^{\updownarrow}$. The crossing number of $H(w, s)$ is established by Corollary 11.

The number of crossings can be decreased after removing any edge from the drawing in Figure 1 (b). This also applies to the generalization of the drawing, thus H_w is a k-degenerate tile. The propagation of the twisted family $\mathcal{F}'$ demonstrates $\mathrm{tcr}\left(\otimes(\mathcal{H}(w, s)^{\updownarrow}/i)\right) \geq k$ for any $i \neq s$, thus $\mathcal{H}(w, s)$ is a k-critical sequence. Criticality of $H(w, s)$ follows by Corollary 8. $\square$

3 Staircase strips in tiles

In this section, we study twisted staircase strips. Using these gadgets, we construct new crossing-critical graphs with average degree close to three.
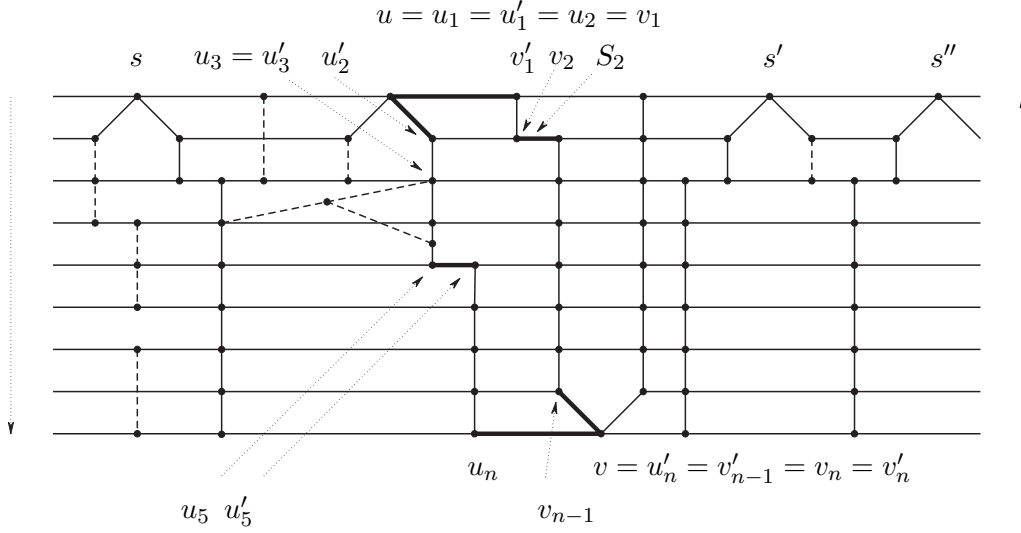
8

Figure 2: A general staircase strip in a tile. Leftmost and rightmost arrows indicate the ordering of the wall vertices. Dashed edges are part of the tile but not of the staircase strip.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a sequence of traversing paths in a tile T with the property $\lambda(P_i) \leq \lambda(P_j)$ and $\rho(P_i) \geq \rho(P_j)$ for $i < j$. Assume that they are pairwise disjoint, except for the pairs P_1, P_2 and P_{n-1}, P_n, which may share vertices, but not edges. For $u \in V(P_1) \cap V(P_2)$ and $v \in V(P_{n-1}) \cap V(P_n)$, we say that u is *left* of v (cf. Figure 2) if there exist internally disjoint paths Q_u and Q_v from u to v such that:

(s.i) there exist vertices $u_1, u_1', \ldots, u_n, u_n'$ that appear in this order on Q_u,

(s.ii) there exist vertices $v_1, v_1', \ldots, v_n, v_n'$ that appear in this order on Q_v,

(s.iii) $u = u_1 = u_1' = u_2 = v_1$ and $v = u_n' = v_{n-1}' = v_n = v_n'$,

(s.iv) $v_1', v_2, v_2', u_2' \notin P_1 \cap P_2$ and $v_{n-1}, u_{n-1}, u_{n-1}', u_n \notin P_{n-1} \cap P_n$,

(s.v) for $i = 1, \ldots, n$, $R_i := u_i P_i u_i' \subseteq P_i \cap Q_u$, with equality for $i \neq n - 1$,

(s.vi) for $i = 1, \ldots, n$, $S_i := v_i P_i v_i' \subseteq P_i \cap Q_v$, with equality for $i \neq 2$,

(s.vii) $R_{n-1} = (P_{n-1} \cap Q_u) - R_n$ and $S_2 = (P_2 \cap Q_v) - S_1$,

(s.viii) if $'u, u' \in P_1 \cap P_2$ are two vertices with $v_1' \in {}'u P_1 u'$, then $v_2 \in {}'u P_2 u'$,

(s.ix) if $'v, v' \in P_{n-1} \cap P_n$ are two vertices with $u_n \in {}'v P_n v'$, then $u_{n-1}' \in {}'v P_{n-1} v'$, and

(s.x) $\lambda(P_i) u_i u_i' v_i v_i' \rho(P_i)$ lie in this order on P_i for $i = 1, \ldots, n$.

Similarly, we define when u is *right* of v. We say that $\mathcal{P}$ forms a *twisted staircase strip* of width n in the tile T if the vertex u is either left or right of the vertex v whenever $u \in V(P_1) \cap V(P_2)$ and $v \in V(P_{n-1}) \cap V(P_n)$.

9

Vertex u in Figure 2 is left of v. The features establishing this fact are emphasized. The subpaths $u_i'Q_u u_{i+1}$ and $v_i'Q_v v_{i+1}$ are, for $i = 2, \ldots, n-1$, internally disjoint from P_j by (s.v) and (s.vi), for any $j = 1, \ldots, n$, and their length is at least one. They are represented by solid vertical edges in the figure. However, the length of R_i and S_i, $i = 1, \ldots, n$, may be zero; the thick edges in the figure emphasize the instances when their length is positive. Solid edges in Figure 2 are part of a twisted staircase strip, dashed edges are not. Note that the vertices u and s are left of v and that the vertices s' and s'' are right of v.

Theorem 13 *Let T be a tile and assume that $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ forms a twisted staircase strip of width n in T. Then, $\mathrm{tcr}(T) \geq \binom{n}{2} - 1$.*

Proof. If a wall vertex v in a tile T has degree d, then the tile crossing number of T is not changed if d new neighbors $v_1, \ldots, v_d$ of degree one are attached to v and v is in its wall replaced by $v_1, \ldots, v_d$. Thus, we may assume that all paths in $\mathcal{P}$ have distinct startvertices in λ and distinct endvertices in ρ.

Let D be any optimal tile drawing of T. By Lemma 10, there are at least $\binom{n}{2} - 2$ crossings in D, since the set $\mathcal{F} = \{\{P_i, P_j\} \mid 1 \leq i < j \leq n\} \setminus \{\{P_1, P_2\}, \{P_{n-1}, P_n\}\}$ is a twisted family in T. For $\{P_i, P_j\} \in \mathcal{F}$, let P_i cross P_j at $x_{i,j}$. In what follows, we contradict the assumption

$$x_{i,j} \text{ are all the crossings of } D. \tag{3.2}$$

For $i = 1, \ldots, n$, let P_i be oriented from $\lambda(P_i)$ to $\rho(P_i)$. The assumption (3.2) implies that the induced drawing of every P_i is a simple curve. This curve splits the unit square $\Delta = I \times I$ containing D into two disjoint open disks, the lower disk Δ_i^- bordering $[0,1] \times \{0\}$ and the upper disk Δ_i^+ bordering $[0,1] \times \{1\}$.

Claim 1: *At $x_{i,j}$, the path P_j crosses from Δ_i^- into Δ_i^+ and the path P_i crosses from Δ_j^+ into Δ_j^-.* This follows from $i < j$ and the orientation of paths P_i and P_j.

As $\lambda(P_2) \in \Delta_1^-$ and $\rho(P_2) \in \Delta_1^+$, there is a vertex $u \in V(P_1) \cap V(P_2)$ where P_2 crosses P_1 from Δ_1^- to Δ_1^+. Also, there is a vertex $v \in V(P_{n-1}) \cap V(P_n)$, such that P_{n-1} crosses from Δ_n^+ into Δ_n^- at v. Then Claim 1 holds for $x_{1,2} = u$ and $x_{n-1,n} = v$.

By symmetry, we may assume that u is left of v in T. Let Q_u and Q_v be the corresponding paths in T. P_2 enters Δ_1^+ at u, and (3.2), (s.iii), (s.iv), and (s.v) imply $u_2' \in \Delta_1^+$. Similarly, $v_{n-1} \in \Delta_n^+$ by (3.2), (s.iii), (s.iv), and (s.vi).

Claim 2: *If any point y of $w_i'Q_w$ lies in Δ_i^- for $w \in \{u, v\}$ and $i \in \{1, \ldots, n\}$, $w_i \neq u_{n-1}$, then the path Q_w must at w_i' enter Δ_i^-. If $w \neq u$ or $i \neq n-1$, the segment $w_i'Q_w y$ does not cross from Δ_i^+ to Δ_i^- due to Claim 1, thus it must lie in Δ_1^-.*

Claim 3: *If there is a point y of $Q_w w_i$ in Δ_i^- for $w \in \{u, v\}$ and $i \in \{1, \ldots, n\}$, $w_i \neq v_2$, then Q_w must at w_i leave Δ_i^-.* Otherwise, the segment $yQ_w w_i$ would contradict Claim 1 at x_{ji} for some $j < i$.

Claim 4: *For $3 \leq i \leq n$, neither of u_i, v_i lies in Δ_1^-.* Assume some $u_i \in \Delta_1^-$. As $u_2' \in \Delta_1^+$, the path Q_u would contradict Claim 1 at $x_{1,j}$ for some j, $1 < j < i$. Assume $v_i \in \Delta_1^-$. Due to the orientation of P_i, (3.2), and (s.x), $u_i \in \Delta_1^-$, a contradiction.

Claim 5: *For $1 \leq i \leq n-2$, neither of u_i', v_i' lies in Δ_n^-.* Assume $v_i' \in \Delta_n^-$. As $v_{n-1} \in \Delta_n^+$, the path Q_v would contradict Claim 1 at $x_{j,n}$ for some j, $i < j < n$. To complete the proof, observe that if $u_i' \in \Delta_n^-$ then $v_i' \in \Delta_n^-$ by (3.2) and (s.x).

In what follows, we prove that the subdrawing of D induced by $Q_u \cup Q_v \cup (\bigcup_i P_i)$ contains a *new* crossing, distinct from $x_{i,j}$, which contradicts (3.2). We first simplify the subdrawing

and obtain a drawing D' in which for every i, j, $1 \le i < j \le n$, the paths P_i and P_j share precisely one point. We use the following steps:

- All vertices of $P_1 \cap P_2$, $P_{n-1} \cap P_n$ at which the two paths do not cross are split.

- As D is a tile drawing, there is an even number of crossing vertices in $V(P_1) \cap V(P_2)$ preceding u on P_1. For a consecutive pair x, y of such vertices, the paths P_1 and P_2 are uncrossed by rerouting $x P_1 y$ along $x P_2 y$ and vice versa. The vertices x and y are split afterwards. The segments of P_{n-1} and P_n following v are uncrossed in a similar manner. By (s.i), (s.ii), (s.iii), and (s.x), the paths Q_u and Q_v are not affected.

- For any pair of vertices of $S_1 \cap P_2 - \{u\}$, the paths P_1 and P_2 are uncrossed in the same way. Due to (s.v), the vertex u'_2 is not on any of the two affected segments. Due to (s.iv), (s.vi) and (s.viii), neither of the segments can contain v'_1, v_2 or v'_2. Thus, $u'_2, v_2, v'_2 \in P_2$ and $v'_1 \in P_1$ after the uncrossing. As all the pairs can be uncrossed, we may assume there is at most one crossing vertex in $S_1 \cap P_2$ distinct from u. But existence of such vertex implies by (s.viii) that $v_2 \in \Delta_1^-$, further implying by (s.vi) and (s.vii) that $v'_2 \in \Delta_1^-$. By (3.2), the segment $v'_2 Q_v v_3$ does not cross P_1, thus v_3 lies in Δ_1^-, contradicting Claim 4.

- As in the previous step, the paths P_{n-1} and P_n are uncrossed at any pair of vertices of $R_n \cap P_{n-1}$. Existence of a single remaining crossing vertex in $R_n \cap P_{n-1}$ would by (s.iv), (s.v), (s.vii), and (s.ix) imply $u'_{n-2} \in \Delta_n^-$, violating Claim 5.

- As D' is a tile drawing, there is an even number of crossing vertices in $v'_1 P_1 \cap P_2$. By (s.viii) and (s.x), uncrossing the paths P_1, P_2 as before does not affect Q_v. Similarly, uncrossing the paths P_{n-1} and $P_n u_n$ does not affect Q_u due to (s.ix) and (s.x).

All crossings in thus obtained drawing D' are also crossings of D, but some crossings of P_1 with P_i may have become crossings of P_2 and P_i and vice versa. The same applies to the pair (P_{n-1}, P_n). We replace the labels $x_{i,j}$ accordingly. Until the end of the proof, we are concerned with the drawing D' only. In the new drawing, Claim 2 holds for $w_i = u_{n-1}$, Claim 3 for $w_i = v_2$, Claim 4 for $i = 2$, and Claim 5 for $i = n - 1$.

Claim 6: *For $1 \le i < j \le n$, the subpath R_i of Q_u does not cross the subpath S_j of Q_v at $x_{i,j}$.* Suppose it does and take the maximal such i. By Claim 1 and (s.x), u_j and v_j lie in Δ_i^-. Claim 2 implies that Q_u and Q_v enter Δ_i^- at u'_i and v'_i. Similarly, u'_i and v'_i lie in Δ_j^- and Claim 3 implies that Q_u and Q_v leave Δ_j^- at u_j, v_j. Thus, the segments $u'_i Q_u u_j$ and $v'_i Q_v v_j$ lie in the intersection $\Delta' = \Delta_i^- \cap \Delta_j^-$. Δ' is a disk as P_i and P_j do not self-cross and cross each other only once. The vertices u'_i, v'_i, u_j, v_j lie in this order on the boundary of Δ', so the segments must intersect in Δ'. This contradicts either the assumption (3.2) or the maximality of i. Claim 6 follows.

Let γ^u denote the simplified path $P_1 u Q_u v P_{n-1}$: whenever this path self-crosses, the circuit is shortcut. Let γ_1^u, γ_2^u, and γ_3^u be the (possibly empty) segments of γ^u corresponding to P_1, Q_u, and P_{n-1}. Similarly, let γ^v denote the simplified path $P_2 u Q_v v P_n$ with the segments γ_1^v, γ_2^v, and γ_3^v. Using the induced orientation of γ^u and γ^v, we define disks Δ_u^+, Δ_u^-, Δ_v^+, and Δ_v^- to be the respective lower and upper disks. The endvertices of γ^u and γ^v interlace in the boundary of $[0, 1] \times [0, 1]$, thus these paths must cross at some crossing $z = z_{i,j}$ of segments
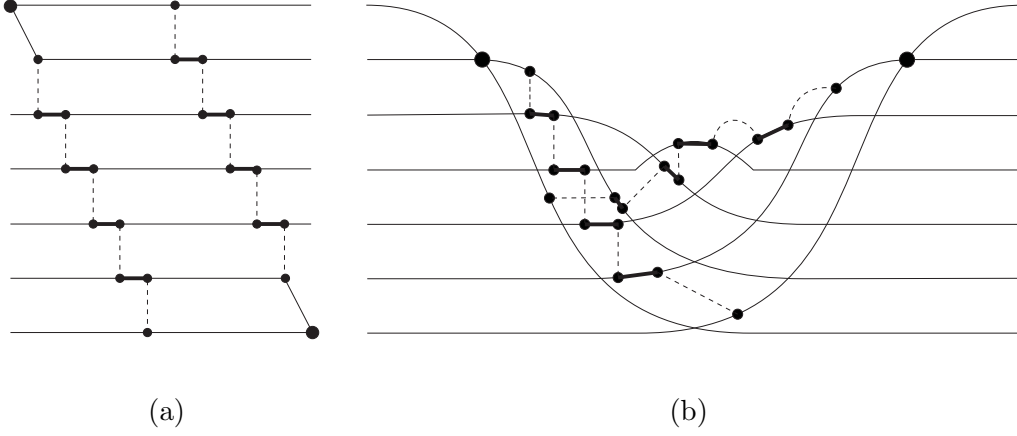
Figure 3: (a) The tile S_7. (b) A tile drawing of S_7 with 20 crossings.

γ_i^u and γ_j^v. We contradict the assumption that $z = x_{i,j}$ for some i, j. Due to the definition of γ^u and γ^v, there are nine possibilities for z:

(1) $z = z_{1,1} = u$ is a touching of γ^u and γ^v.

(2) $z = z_{1,2} = x_{1,i}$ for some $i > 2$. Thus, $v_i \in \Delta_1^-$ contradicts Claim 4.

(3) $z = z_{1,3} = x_{1,n}$ implies $u_n \in \Delta_1^-$.

(4) $z = z_{2,1} = x_{i,2}$ for some $i > 2$, then $u_i \in \Delta_1^-$.

(5) $z = z_{2,2} = x_{i,j}$ is a crossing of S_i and R_j. Claim 6 implies that $1 \le i < j \le n$. Choose smallest such i and then smallest j. Q_v starts in Δ_u^- and since z is the first crossing of Q_v with γ^u (or one of the other eight cases would apply), Q_v leaves Δ_u^- and enters Δ_u^+ at z. As the orientation of γ^u is aligned with the orientation of R_j, S_i leaves Δ_j^-, which contradicts Claim 1.

(6) $z = z_{2,3} = x_{i,n}$ for some $i < n$, then $u_i' \in \Delta_n^-$, which contradicts Claim 5.

(7) $z = z_{3,1} = x_{2,n-1}$ implies $v_2' \in \Delta_n^-$.

(8) $z = z_{3,2} = x_{i,n-1}$ is the crossing of P_{n-1} and S_i, then $v_i' \in \Delta_n^-$.

(9) $z = z_{3,3} = v$ is a touching of γ^u and γ^v.

Thus, γ^u and γ^v must cross at a new crossing and the statement of the theorem follows. □

The reader shall have no difficulty rigorously describing the tile S_n, $n \ge 3$, an example of which is for $n = 7$ presented in Figure 3 (a). A *staircase tile* of width $n \ge 3$ is a tile obtained from S_n by contracting some (possibly zero) thick edges of S_n. Such a tile is a perfect planar tile. A *staircase sequence* of width n is a sequence of tiles of odd length in which staircase tiles of width n alternate with inverted staircase tiles of width n. Any staircase sequence is a cyclically-compatible sequence of tiles.

Proposition 14 *Let $\mathcal{T}$ be a staircase sequence of width n and odd length $m \geq 4\binom{n}{2} - 5$. The graph $G = \circ(\mathcal{T}^{\updownarrow})$ is a crossing-critical graph with $\mathrm{cr}(G) = \binom{n}{2} - 1$.*

Proof. A generalization of the drawing in Figure 3 demonstrates that $\mathrm{tcr}(S_n) \leq \binom{n}{2} - 1$. As m is odd, the cut $\mathcal{T}^{\updownarrow}/i$ contains a twisted staircase strip of width n for any $i = 0, \ldots, m-1$, and Theorem 13 implies $\mathrm{tcr}(\mathcal{T}^{\updownarrow}/i) \geq \binom{n}{2} - 1$. Planarity of tiles S_n and Lemma 2 establish equality and Corollary 7 implies $\mathrm{cr}(G) = \binom{n}{2} - 1$.

After removing any edge from S_n, we can decrease the number of crossings in the drawing in Figure 3 (b). Thus, S_n is a $\left(\binom{n}{2} - 1\right)$-degenerate tile and $\mathcal{T}$ is a $\left(\binom{n}{2} - 1\right)$-critical sequence; the criticality of G follows by Corollary 8. □

Let S'_n be the inverted tile S_n. Let $\mathcal{S}_{n,m}$ be the staircase sequence $(S_n, S'_n, S_n, S'_n, \ldots, S_n)$ of odd length $m \geq 1$ and $\mathcal{S}(n, m, c)$ the set of graphs obtained from $\circ(\mathcal{S}^{\updownarrow}_{n,m})$ by contracting c thick edges in the tiles of $\mathcal{S}_{n,m}$. These graphs almost settle Question 1:

Proposition 15 ([3, 4]) *Let $r = 3 + \frac{a}{b}$ with $1 \leq a < b$. If $a + b$ is odd, then, for $n \geq \max\left(\frac{5b-a}{2(b-a)}, \frac{7a+b}{4a}, 4\right)$, $m(t) = (2t+1)(a+b)$, and $c(t) = (2t+1)((4n-7)a - b)$, the family $\mathcal{Q}(a, b, n) = \bigcup_{t=n^2}^{\infty} \mathcal{S}(n, m(t), c(t))$ contains $\left(\binom{n}{2} - 1\right)$-crossing-critical graphs with average degree r.*

Demanding the average degree of the graphs in $\mathcal{S}(m, n, c)$ to be $r = 3 + \frac{a}{b}$, $1 \leq a < b$, $a + b$ even, forces $m(t)$ to be an even number and the resulting graphs are no longer critical.

4 Zip product and criticality of graphs

Zip product is an operation on graphs or their drawings that was used in [1, 2] to establish the crossing number of Cartesian products of several graphs with trees. For two graphs G_i $(i = 1, 2)$, their vertices v_i of degree d not incident with multiple edges (we call such vertices *simple*), and a bijection $\sigma : N_1 \to N_2$ of the neighborhoods N_i of v_i in G_i, the *zip product* of the graphs G_1 and G_2 according to σ is the graph $G_1 \odot_{\sigma} G_2$, obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ after adding the edge $u\sigma(u)$ for every $u \in N_1$. We call σ a *zip function* of the graphs. Let $G_1 \,_{v_1}\!\odot_{v_2} G_2$ denote the set of all pairwise nonisomorphic graphs, obtained as a zip product $G_1 \odot_{\sigma} G_2$ for some bijection $\sigma : N_1 \to N_2$.

A drawing D_i of a graph G_i imposes a cyclic ordering of the edges incident with v_i, which defines a labeling $\pi_i : N_i \to \{1, \ldots, d\}$ up to a cyclic permutation. A *zip function* of the drawings D_1 and D_2 at vertices v_1 and v_2 is $\sigma : N_1 \to N_2$, $\sigma = \pi_2^{-1}\pi_1$.

Lemma 16 ([1]) *For $i = 1, 2$, let D_i be an optimal drawing of G_i, let $v_i \in V(G_i)$ be a simple vertex of degree d, and let σ be a zip function of D_1 and D_2 at v_1 and v_2. Then, $\mathrm{cr}(G_1 \odot_{\sigma} G_2) \leq \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$.*

Let $v \in V(G)$ be a vertex of degree d in G. A *bundle* of v is a set B of d edge disjoint paths from v to some vertex $u \in V(G)$, $u \neq v$. Two bundles B_1 and B_2 of v are *coherent* if the sets of edges $E(B_1) \cap E(G - v)$ and $E(B_2) \cap E(G - v)$ are disjoint.

13

Lemma 17 ([2]) *For $i = 1, 2$, let G_i be a graph, $v_i \in V(G_i)$ its simple vertex of degree d, and $N_i = N_{G_i}(v_i)$. Also assume that v_i has two coherent bundles in G_i. Then, $\mathrm{cr}(G_1 \odot_\sigma G_2) \geq \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$ for any bijection $\sigma : N_1 \to N_2$.*

The following observations are useful in iterative applications of the zip product.

Lemma 18 *Let G_1 and G_2 be disjoint graphs, $v_i \in V(G_i)$ simple, $\deg_{G_i}(v_i) = d$, and $G \in G_1 \; {}_{v_1}\odot_{v_2} \; G_2$.*

   (i) *If v_2 has a bundle in G_2 and $v \in V(G_1)$ has k pairwise coherent bundles in G_1, then v has k pairwise coherent bundles in G.*

   (ii) *If, for $i = 1, 2$, the graph G_i is k_i-connected, $k_i \geq 2$, then G is k-connected for $k = \min(k_1, k_2)$.*

   (iii) *If, for $i = 1, 2$, the graph G_i is k_i-edge-connected, $k_i \geq 2$, then G is k-edge-connected for $k = \min(k_1, k_2)$.*

Proof. **(i)**: See [2].

   (ii): Let $S \subseteq V(G)$ be a separator of G. If $S \subseteq V(G_i - v_i)$, then, as $G_{3-i} - v_{3-i}$ is nonempty and $(k_{3-i} - 1)$-connected, S is a separator in G_i and $|S| \geq k$. Let $S_i = S \cap V(G_i - v_i)$ and $S_i \neq \emptyset$ for $i = 1, 2$. If $S_i \cup \{v_i\}$ is a separator in G_i for one of $i = 1, 2$, then $|S| \geq k$. Otherwise, the vertices of $G_i - v_i - S$ are all in the same component of $G - S$ for both $i = 1, 2$, thus $|S| \geq d \geq k$.

   (iii): The argument is similar to (ii). $\qquad\square$

Let $S \subset V(G)$ be a set and $\Gamma \subseteq \mathrm{Aut}(G)$ a group. We say that S is Γ-*homogeneous* in G if any permutation π of S can be extended to an automorphism $\sigma \in \Gamma$. For $S \subseteq V(G)$, let $\Gamma(S)$ be the pointwise stabilizer of S in $\mathrm{Aut}(G)$. We say that a vertex $v \in V(G)$ has a *homogeneous neighborhood* in G if $N_G(v)$ is $\Gamma(\{v\})$-homogeneous in G.

If all the vertices in $N_G(v)$ have the same set of neighbors for a vertex $v \in V(G)$, then v has a homogeneous neighborhood G. Thus, every vertex of a complete or complete bipartite graph K has such neighborhood in K.

Lemma 19 *For $i = 1, 2$, let G_i be a graph with a simple vertex $v_i \in V(G_i)$ of degree d. If $d = 3$ or v_2 has a homogeneous neighborhood in G_2, then $\mathrm{cr}(G) \leq \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$ for every $G \in G_1 \; {}_{v_1}\odot_{v_2} \; G_2$.*

Proof. Assume $N_1 = N_{G_1}(v_1)$, $N_2 = N_{G_2}(v_2)$, and let the zip function of G be $\sigma : N_1 \to N_2$. For $i = 1, 2$, let D_i be an optimal drawing of G_i and let $\pi_i : N_i \to N_i$ denote the vertex rotation around v_i in D_i. For $d > 3$, there exists an automorphism $\rho \in \Gamma_{G_2}(\{v_2\})$ with $\rho/N_2 = \sigma \pi_1 \sigma^{-1} \pi_2^{-1}$. Applying ρ to D_2 produces a drawing D_2' with vertex rotation $\rho \pi_2 = \sigma \pi_1 \sigma^{-1}$ around v_2. Since $\sigma^{-1}(\rho \pi_2)^{-1} \sigma \pi_1 = \mathrm{id}$, σ is a zip function of D_1 and D_2'. If $d = 3$, then σ is a zip fucntion of D_1 and either D_2 or its mirrored image D_2'. The claim follows by Lemma 16. $\qquad\square$

Lemma 20 *For $i = 1, 2$, let G_i be a graph with a simple vertex $v_i \in V(G_i)$ of degree d. Assume that v_2 has a homogeneous neighborhood in G_2. If, for some vertex $v \in G_1$, $v \neq v_1$, its neighborhood $N = N_{G_1}(v)$ is $\Gamma_{G_1}(\{v, v_1\})$-homogeneous, then v has a homogeneous neighborhood in $G \in G_1 {}_{v_1}\odot_{v_2} G_2$.*

Proof. Assume $N_1 = N_{G_1}(v_1)$, $N_2 = N_{G_2}(v_2)$, and let the zip function of G be $\sigma : N_1 \to N_2$. For a permutation π of N, there exists $\sigma_1 \in \Gamma_{G_1}(\{v, v_1\})$, such that $\sigma_1/N = \pi$. Let $\pi_1 = \sigma_1/N_1$, and set $\pi_2 = \sigma\pi_1\sigma^{-1}$. As v_2 has a homogeneous neighborhood, there exists an automorphism $\sigma_2 \in \Gamma_{G_2}(v_2)$ with $\sigma_2/N_2 = \pi_2$. It is easy to verify that a function $\Phi : G \to G$ with $\Phi/(G_i - v_i) = \sigma_i/(G_i - v_i)$ for $i = 1, 2$, is an automorphism of $\Gamma_G(v)$, for which $\Phi/N = \pi$. Thus, v has a homogeneous neighborhood in G. $\square$

Theorem 21 *For $i = 1, 2$, let G_i be a k_i-crossing-critical graph with a simple vertex $v_i \in V(G_i)$ of degree d. If $d \neq 3$, then let v_i have a homogeneous neighborhood. If $\mathrm{cr}(G) \geq k$ for $k = \max\{\mathrm{cr}(G_i) + k_{3-i} \mid i \in \{1, 2\}\}$ and $G \in G_1 {}_{v_1}\odot_{v_2} G_2$, then G is k-crossing-critical.*

Proof. Again, assume $N_1 = N_{G_1}(v_1)$, $N_2 = N_{G_2}(v_2)$, and let the zip function of G be $\sigma : N_1 \to N_2$. Let $e \in E(G)$, and assume $e \in E(G_1 - v_1)$. Let D_1 be an optimal drawing of $G_1 - e$ and D_2 an optimal drawing of G_2. We adjust D_2 either using an appropriate automorphism in $\Gamma_{G_2}(\{v_2\})$ for $d > 3$ or mirroring for $d = 3$ similarly as in the proof of Lemma 19 and combine D_2 with D_1 to produce a drawing of $G - e$ with at most k crossings. Similar arguments apply for $e \in E(G_2 - v_2)$.

If $e = v\sigma(v)$ for $v \in N_1$, let D_1 be an optimal drawing of $G_1 - vv_1$ and D_2 an optimal drawing of $G_2 - v_2\sigma(v)$. If $d = 3$, we can clearly combine D_1 and D_2 into a drawing of G with at most k crossings. Otherwise, let $\pi_i : N_i \to N_i$ be the vertex rotation around v_i in D_i and $\rho \in \Gamma_{G_2}(\{v_2\})$ an automorphism of G_2 with $\rho/(N_2 \setminus \{\sigma(v)\}) = \sigma\pi_1\sigma^{-1}\pi_2^{-1}$. The vertices of N_2 can be rearranged with ρ as in the proof of Lemma 19, thus $G - e$ can be drawn with at most $k_1 + k_2$ crossings. $\square$

Lemma 17 states that two coherent bundles at each v_i are a sufficient condition for $\mathrm{cr}(G) \geq k$ in Theorem 21.

Argument of Theorem 21 has a generalization to (not necessarily critical) graphs that have a special vertex cover. Let G be a graph and $S = \{v_1, \ldots, v_t\} \subseteq V(G)$. For each $v_i \in S$, let G_i be a graph and let $u_i \in V(G_i)$ be a simple vertex of degree $d(u_i) = d(v_i)$ having two coherent bundles in G_i. Let $\mathcal{S} := \{(v_i, G_i) \mid i \in \{1, \ldots, t\}\}$. The family $G^{\mathcal{S}} := \Gamma^t$ is defined inductively as follows: $\Gamma^0 = \{G\}$, and, for $i = 1, \ldots, t$, let $\Gamma^i := \bigcup_{H \in \Gamma^{i-1}} H {}_{v_i}\odot_{u_i} G_i$. Further, let $\mathcal{S}_i := \mathcal{S} \setminus \{(v_i, G_i)\}$.

Theorem 22 *Let G be a graph, S be its vertex cover consisting of simple vertices of degree three each having two coherent bundles, and $\mathcal{S}$ be defined as above. If each graph G_i is k_i-crossing-critical for $i = 1, \ldots, t$, then every $\bar{G} \in G^{\mathcal{S}}$ is k-crossing-critical for $k = \max\{\mathrm{cr}(\bar{G}) - \mathrm{cr}(G_i) + k_i \mid i \in \{1, \ldots, t\}\}$ and has crossing number $\mathrm{cr}(\bar{G}) = \mathrm{cr}(G) + \sum_{i=1}^{t} \mathrm{cr}(G_i)$.*

Proof. Iterative application of Lemmas 17, 18 (i), and 19 implies $\mathrm{cr}(\bar{G}) = \mathrm{cr}(G) + \sum_{i=1}^{t} \mathrm{cr}(G_i)$. To establish criticality of $\bar{G}$, let $e \in E(G^{\mathcal{S}})$ be an arbitrary edge and let $\bar{G}_j \in G^{\mathcal{S}_j}$, $j = 1, \ldots, t$, be the graph, such that $\bar{G} \in \bar{G}_j {}_{v_j}\odot_{u_j} G_j$.
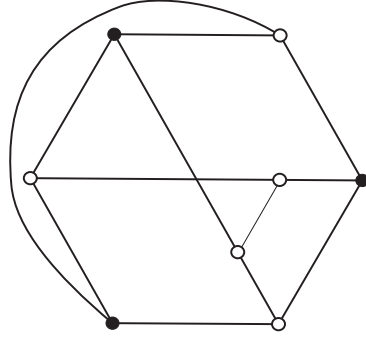
Figure 4: Graph with a vertex cover of cubic vertices each having two coherent bundles.

Case 1: Assume $e \in E(G_j - v_j)$ for some $j \in \{1, \ldots, t\}$. Let D_1 be an optimal drawing of $\bar{G}_j$ with v_j in the infinite face and let D_2 be an optimal drawing of $G_j - e$ with u_j in the infinite face. We can combine $D_1 - v_j$ and $D_2 - u_j$ into a drawing D of $G - e$. By Lemma 19, D has at most $\mathrm{cr}(G) + k_j + \sum_{i \neq j} \mathrm{cr}(G_i) \leq k$ crossings.

Case 2: Assume $e \notin E(G_i - v_i)$ for any $i \in \{1, \ldots, t\}$. As S is a vertex cover in G, there exists $j \in \{1, \ldots, t\}$, such that e connects some neighbor x of $u_j \in V(G_j)$ with some neighbor y of v_j in $\bar{G}_j$. Let $e_1 = v_j y \in E(\bar{G}_j)$, $e_2 = u_j x \in E(G_j)$, and let D_1 be an optimal drawing of $\bar{G}_j - e_1$ with v_j on the infinite face and D_2 an optimal drawing of $G_j - e_2$ with u_j in the infinite face. We can combine $D_1 - v_j$ and $D_2 - u_j$ into a drawing D of $G - e$. By Lemma 19, D has at most $\mathrm{cr}(G - e) + k_j + \sum_{i \neq j} \mathrm{cr}(G_i) \leq k$ crossings. $\square$

Leaños and Salazar established a decomposition of 2-connected crossing-critical graphs into smaller 3-connected crossing-critical graphs in [16]. Theorem 22, in combination with the graph in Figure 4, which has a vertex cover consisting of cubic vertices with two coherent bundles but is not crossing-critical, suggests that a similar decomposition does not exist for 3-connected crossing-critical graphs.

For $d, d' \geq 3$, let $K_{d,d'}$ be a properly 2-colored complete bipartite graph: vertices of degree d are colored black and vertices of degree d' are colored white. For $p \geq 1$, let the family $\mathcal{R}(d, d', p)$ consist of graphs with 2-colored vertices, obtained as follows: $\mathcal{R}(d, d', 1) = \{K_{d,d'}\}$ and $\mathcal{R}(d, d', p) = \bigcup_{G \in \mathcal{R}(d,d',p-1)} G_{v_1} \odot_{v_2} K_{d,d'}$, where v_1 (respectively, v_2) is a black vertex in G ($K_{d,d'}$). If $d = d' = 3$, we allow v_i to be any vertex. We preserve the colors of vertices in the zip product, thus the graphs in $\mathcal{R}(d, d', p)$ are not properly colored for $p \geq 2$.

Proposition 23 *Let $d, d' \geq 3$. Then every graph $G \in \mathcal{R}(d, d', p)$ is a simple 3-connected crossing-critical graph with $\mathrm{cr}(G) = p \, \mathrm{cr}(K_{d,d'})$.*

Proof. By induction on p and using Lemma 20, we show that all black vertices of G have homogeneous neighborhoods. Iterative application of Lemmas 17, 18 (i), 19, and Theorem 21 establish the crossing number of G and its criticality. $\square$

Jaeger proved the following result:

Theorem 24 ([13]) *Every 3-connected cubic graph with crossing number one has chromatic index three.*

Graphs of the family $\mathcal{R}(3,3,p)$ in a zip product with Petersen graph show that a similar result cannot be obtained for any crossing number greater than one.

Proposition 25 ([3]) *For $k \geq 2$, there exist simple cubic 3-connected crossing-critical graphs with crossing number k and with no 3-edge-coloring.*

5   The main construction

Theorem 26 *Let $r \in (3,6)$ be a rational number and k an integer. There exists a convex continuous function $f : (3,6) \to \mathbb{R}^+$ such that, for $k \geq f(r)$, there exists an infinite family of simple 3-connected crossing-critical graphs with average degree r and crossing number k.*

Proof. We present a constructive proof for

$$f(r) = 240 + \frac{512}{(6-r)^2} + \frac{224}{6-r} + \frac{25}{16(r-3)^2} + \frac{40}{r-3}.$$

A sketch of the construction is as follows: The graphs are obtained as a zip product of crossing-critical graphs from the families $\mathcal{S}$ and $\mathcal{R}$, and of the graphs H, all defined above. The graphs H allow average degree close to six and the graphs from $\mathcal{S}$ allow average degree close to three. A disjoint union of two such graphs consisting of a proportional number of tiles would have a fixed average degree and crossing number. The zip product compromises the pattern needed for fixed average degree, for which we compensate with the graphs from $\mathcal{R}$. Their role is also to fine-tune the desired crossing number of the resulting graph.

More precisely, let $\Gamma(n,m,c,w,s,p,q)$ be the family of graphs, constructed in the following way: first we combine $G_1 \in \mathcal{S}(n,m,c)$ and $G_2 = H(w,s)$ in the family $\Gamma(n,m,c,w,s,0,0) = \bigcup_{G_1,G_2} \bigcup_{v_1,v_2} G_1 \, {}_{v_1}\!\odot_{v_2} G_2$. Further, we combine $G_1 \in \Gamma(n,m,c,w,s,0,0)$ and $G_2 \in \mathcal{R}(3,3,p)$ in the family $\Gamma(n,m,c,w,s,p,0) = \bigcup_{G_1,G_2} \bigcup_{v_1,v_2} G_1 \, {}_{v_1}\!\odot_{v_2} G_2$. Finally, we combine the graphs $G_1 \in \Gamma(n,m,c,w,s,p,0)$ and $G_2 \in \mathcal{R}(3,5,q)$ in the family $\Gamma(n,m,c,w,s,p,q) = \bigcup_{G_1,G_2} \bigcup_{v_1,v_2} G_1 \, {}_{v_1}\!\odot_{v_2} G_2$. In each case, $v_i \in V(G_i)$ is any vertex of degree three, as all such vertices have two coherent bundles. Propositions 12, 14, and 23 imply that the graphs used in construction are crossing-critical graphs whenever the following conditions are satisfied:

$$
\begin{align}
n \;&\geq\; 3, & (5.3)\\
m \;&=\; 2m' + 1, & (5.4)\\
m' \;&\geq\; 2\binom{n}{2}, & (5.5)\\
c \;&\geq\; 0, & (5.6)\\
c \;&\leq\; 2m(n-3), & (5.7)\\
w \;&\geq\; 0, & (5.8)\\
s \;&\geq\; 4(32w^2 + 56w + 31), & (5.9)\\
p \;&\geq\; 1, \text{ and} & (5.10)\\
q \;&\geq\; 1. & (5.11)
\end{align}
$$

Results in [14] establish $\mathrm{cr}(K_{3,5}) = 4$, thus Theorem 21 together with Lemmas 17, 18 (i), and 19 implies that subject to (5.3)–(5.11) the graphs in $\Gamma(n,m,c,w,s,p,q)$ are crossing-

critical with crossing number

$$k = \binom{n}{2} + 32w^2 + 56w + p + 4q + 30. \tag{5.12}$$

Their average degree is

$$\bar{d} = 6 - \frac{4(m'(6n - 11) + 3n + 3p + 3q + 4s - c - 7)}{2m'(4n - 7) + 4n + 4sw + 9s + 4p + 6q - c - 9}. \tag{5.13}$$

Using (5.12) we express p in terms of k and other parameters. We set s and m to be a linear function of a new parameter t, which will determine the size of the resulting graph. We substitute these values into (5.13). Using c we eliminate all the terms in the denominator that are independent of t. Parameter q plays the same role in the numerator. Then t cancels and we set the coefficients of the linear functions to yield the desired average degree. Finally, parameters n, w, and the constant terms of the linear functions are selected to satisfy the constraints (5.3)–(5.11). A more detailed analysis might produce a smaller lower bound f, but one constant term was selected to be zero to simplify the computations.

More precisely, let $r = 3 + \frac{a}{b}$, $0 < a < 3b$, and $k \geq f(r)$. Perform the following integer divisions:

$$
\begin{aligned}
b &= b'a + b_r, \\
b' &= 4b'' + b'_r, \\
4b &= \bar{b}(3b - a) + \bar{b}_r, \text{ and} \\
k - \tfrac{b''(b''+5)}{2} - 8\bar{b}(4\bar{b} + 7) &= k'(2b'' + 5) + k_r.
\end{aligned}
$$

For some integer t set

$$
\begin{aligned}
n &= b'' + 4, \\
m_t &= 2t(27b - 9a - 4\bar{b}_r) - 2k' + 3, \\
c &= 2k' - 12b'' - 6k_r - 33, \\
w &= \bar{b}, \\
s_t &= 2t((4b'' + 9)a - b), \\
p &= k - \left(\tfrac{b''(b''+23)}{2} + 8\bar{b}(4\bar{b} + 7) + 4k_r + 56 \right), \text{ and} \\
q &= 2b'' + k_r + 5.
\end{aligned}
$$

The family $\Gamma(a, b, k) = \bigcup_{t=k}^{\infty} \Gamma(n, m_t, c, w, s_t, p, q)$ is an infinite family of crossing-critical graphs with average degree r and crossing number k. Verification of the constraints (5.3)–(5.11) for any $r \in (3, 6)$ and $k \geq f(r)$ requires some tedious computation that is omitted here; an interested reader can find it in [4]. The function f is a sum of functions that are convex on $(3, 6)$ and thus itself convex. The graphs of $\Gamma(a, b, k)$ are 3-connected by Lemma 18 (ii). □

The convexity of the function f in Theorem 26 implies $N_I = \max\{f(r_1), f(r_2)\}$ is a universal lower bound on k for rational numbers within any closed interval $I = [r_1, r_2] \subseteq (3, 6)$.
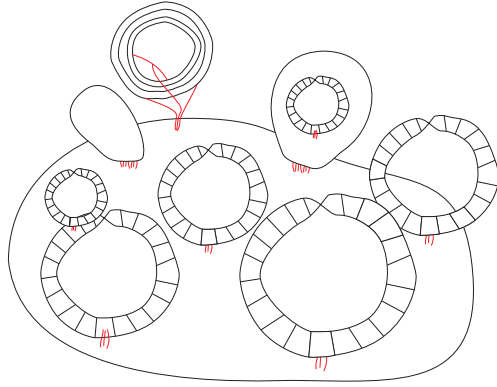
Figure 5: Structure of known large k-crossing-critical graphs.

6   Structure of crossing-critical graphs

It has recently been established that all large 2-crossing-critical graphs are obtained as cyclizations of long sequences, composed out of copies of a small number of different tiles [5, 6]. The construction of crossing-critical graphs using zip product demonstrates that no such classification of tiles can exist for $k \geq 4$: by a generalized zip product of a graph and a tile, as proposed in [3], one can obtain an infinite sequence of k-degenerate tiles, all having the same tile crossing number. These tiles in combination with corresponding perfect planar tiles yield k-crossing-critical graphs.

For k large enough, one can obtain k-crossing-critical graphs from an arbitrary (not necessarily critical) graph that has a vertex cover consisting of simple vertices of degree three with two coherent bundles, cf. Theorem 22. Figure 5 sketches the described structure.

The following questions remain open regarding the degrees of vertices in k-crossing-critical graphs:

Question 27 ([20]) *Do there exist an integer $k > 0$ and an infinite family of (simple) 5-regular 3-connected k-crossing-critical graphs?*

Question 28 *Do there exist an integer $k > 0$ and an infinite family of (simple) 3-connected k-crossing-critical graphs of average degree six?*

Arguments of [20] used to establish that, for $k > 0$, there exist only finitely many k-crossing-critical graphs with minimum degree six extend to graphs with a bounded number of vertices of degree smaller than six. Thus, we may assume that a family positively answering Question 28 would contain graphs with arbitrarily many vertices of degree larger than six. But only vertices of degrees three, four, or six appear arbitrarily often in the graphs of the known infinite families of k-crossing-critical graphs. We thus propose the following question, an answer to which would be a step in answering Questions 27 and 28.

Question 29 *Does there exist an integer $k > 0$, such that, for every integer n, there exists a 3-connected k-crossing-critical graph G_n with more than n vertices of degree distinct from three, four and six?*

We can obtain arbitrarily large crossing-critical graphs with arbitrarily many vertices of degree d, for any d, by applying the zip product to graphs $K_{3,d}$, $K_{d,d}$, and the graphs from the known infinite families. However, the crossing numbers of these graphs grow with the number of such vertices.

7 Acknowledgement

References

[1] D. Bokal, On the crossing number of Cartesian products with paths, J. Combin. Theory Ser. B. 97 (2007), 381–384.

[2] D. Bokal, On the crossing number of Cartesian products with trees, J. Graph Theory 56 (2007) 287–300.

[3] D. Bokal, Structural approach to the crossing number of graphs, Dissertation, University of Ljubljana, 2006.

[4] D. Bokal, Technical details regarding infinite families of crossing-critical graphs with prescribed average degree and crossing number, Mathematica™ notebook (2006), `http://arxiv.org/abs/0909.1939`.

[5] D. Bokal, B. Oporowski, R. B. Richter, G. Salazar, Characterization of 2-crossing-critical graphs I: Low connectivity or no V_{2n}-minor, submitted.

[6] D. Bokal, B. Oporowski, R. B. Richter, G. Salazar, Characterization of 2-crossing-critical graphs II: the tiles, in preparation.

[7] R. Diestel, Graph Theory, Graduate Texts in Mathematics, vol. 173, Springer Verlag, New York, 2000.

[8] Z. Dvořák, B. Mohar, Crossing-critical graphs with large maximum degree, submitted, `http://arxiv.org/PS_cache/arxiv/pdf/0907/0907.1599v1.pdf`.

[9] J. F. Geelen, R. B. Richter, G. Salazar, Embedding grids on surfaces, European J. Combin. 25 (2004), 785–792.

[10] P. Hliněný, Crossing-critical graphs and path-width, in: *Proc. 9th Intl. Symposium on Graph Drawing*, Lecture Notes in Computer Science 2265, Springer Verlag, Berlin, 2001, 102–114.

[11] P. Hliněný, Crossing-critical graphs have bounded path-width, J. Combin. Theory Ser. B 88 (2003), 347–367.

[12] P. Hliněný, New infinite families of almost-planar crossing-critical graphs, Electron. J. Combin. 15 (2008), #R102.

[13] F. Jaeger, Tait's theorem for graphs with crossing number at most one, Ars Combin. 9 (1980), 283–287.

[14] D. J. Kleitman, The crossing number of $K_{5,n}$, J. Combin. Theory Ser. B 9 (1971), 315–323.

[15] M. Kochol, Construction of crossing-critical graphs, Discrete Math. 66 (1987), 311–313.

[16] J. Leaños, G. Salazar, On the additivity of crossing numbers of graphs, J. Knot Theory Ramifications 17 (2008), 1043–1050.

[17] B. Pinontoan, R. B. Richter, Crossing numbers of sequences of graphs I: general tiles, Australas. J. Combin. 30 (2004), 197-206.

[18] B. Pinontoan, R. B. Richter, Crossing numbers of sequences of graphs II: planar tiles, J. Graph Theory 42 (2003), 332–342.

[19] R. B. Richter, C. Thomassen, Intersections of curve systems and the crossing number of $C_5 \square C_5$, Discrete Comput. Geom. 13 (1995), 149–159.

[20] R. B. Richter, C. Thomassen, Minimal graphs with crossing number at least k, J. Combin. Theory Ser. B 58 (1993), 217–224.

[21] G. Salazar, Infinite families of crossing-critical graphs with given average degree, Discrete Math. 271 (2003), 343–350.

[22] M. Lomelí, G. Salazar, Nearly-light cycles in embedded graphs and crossing-critical graphs, J. Graph Theory 53 (2006), 151–156.

[23] G. Salazar, On a crossing number result of Richter and Thomassen, J. Combin. Theory Ser. B 79 (2000), 98–99.

[24] J. Širáň, Crossing-critical edges and Kuratowski subgraphs of a graph, J. Combin. Theory Ser. B 35 (1983), 83–92.

[25] J. Širáň, Infinite families of crossing-critical graphs with a given crossing number, Discrete Math. 48 (1984), 129–132.